Spectral Form Factor

Cynthia Yan
Department of Physics, Stanford University, Stanford, CA 94305
(Dated: June 28, 2020)

Submitted as coursework for PH470, Stanford University, Spring 2020

In the paper Black Holes and Random Matrices[3], the authors compared the $g(t)$ graphs of the SYK model and the Random Matrix Theory and find a similarity when t is large. Because there was an established correspondence between large AdS black hole and the SYK model, they concluded that large AdS black holes is a quantum chaotic system.

I. INTRODUCTION

A natural way to characterize the energy spectrum of a Hamiltonian H is to consider the thermal correlator of a Hermitian operator O

$$\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H/2}O(t)e^{-\beta H/2}O(0)\right) = \frac{1}{Z}\sum_{n,m} e^{-(\frac{\beta}{2}+it)E_n}e^{-(\frac{\beta}{2}-it)E_m}\,|\,\langle n|O|m\rangle\,|^2 \quad (1)$$

where Z is the partition function β is inverse temperature, and t is time. But according to the Eigenstate Thermalization Hypothesis[1], the contribution to the correlator from the specific operator O varies smoothly as time becomes large, and we are more interested in the phases that cause oscillations. To ease computation and to gain more physical intuition, with the analytical continuation of the partition function given by

$$Z(\beta,t) = \mathrm{Tr}(e^{-\beta H - iHt}) \quad (2)$$

we can define spectral form factor as

$$g(t;\beta) = \left|\frac{Z(\beta,t)}{Z(\beta)}\right|^2 = \frac{1}{Z(\beta)^2}\sum_{m,n} e^{-\beta(E_m+E_n)}e^{i(E_m-E_n)t} \quad (3)$$

In the paper Black Holes and Random Matrices[3], the authors compared the $g(t)$ graphs of the Sachdev-Ye-Kitaev (SYK) model and the Random Matrix Theory (RMT)[4] and find a similarity when t is large. Because there was an established correspondence between large AdS black hole and the SYK model, they concluded that large AdS black holes is a quantum chaotic system. In this report, we present some important features of the spectral form factor and some physical intuitions behind those.

II. SPECTRAL FORM FACTOR OF SYK

The SYK model was shown to have some features of a gravity dual[5], and it has been recently a convenient simple condensed matter model to study features of gravity and black holes. By studying the spectral form factor of SYK, we can make statements about the Hilbert space of black holes. The SYK Hamiltonian is given by

$$H = \frac{1}{4!}\sum_{a,b,c,d} J_{abcd}\psi_a\psi_b\psi_c\psi_d \quad (4)$$

where ψ_i are Majorana fermions and J are random couplings satisfying a Gaussian distribution. In general we can have interactions between any number q of fermions, but $q = 4$ is the most common choice. In this context, the spectral form factor, its disconnected piece and its connected piece are given by

$$g(t,\beta) = \frac{\langle Z(\beta,t)Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2} \quad (5)$$

$$g_d(t,\beta) = \frac{\langle Z(\beta,t)\rangle_J \cdot \langle Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2} \quad (6)$$

$$g_c(t,\beta) = g(t,\beta) - g_d(t,\beta) \quad (7)$$

The spectral form factor usually needs to be averaged over an ensemble in order to smooth out the curve. Shown in figure 1 is one realization of SYK $g(t)$ and shown in figure 2 is an averaged $g(t)$ over 90 samples. The single realization is erratic but we still see the same trend as in the more smooth curve. As shown in figure 2, the $g(t)$ graph consists of a slope, a dip, a ramp, and a plateau.

III. SPECTRAL FORM FACTOR OF RMT

In RMT, a single realization of matrix M corresponds to a single realization of the Hamiltonian H in the SKY model, so we can write the partition function as instead

$$Z(\beta,t) = \mathrm{Tr}(e^{-\beta M - iMt}) \quad (8)$$

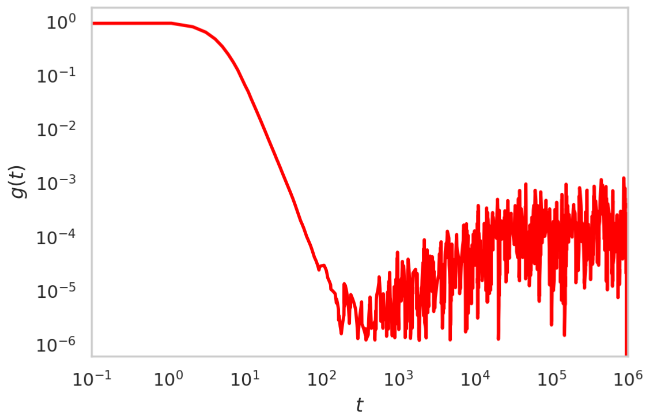

FIG. 1: An illuatration of[3] figure 10. The red curve is $g(t)$ of one sample of SYK and the blue curve is $g(t)$ of many samples of SYK averaged. Source: C. Yan, adapted from the red curve in figure 10 of Cotler, J.S te al[3]. The figure shows one sample of SYK model with $N = 34$ and $\beta = 5$.

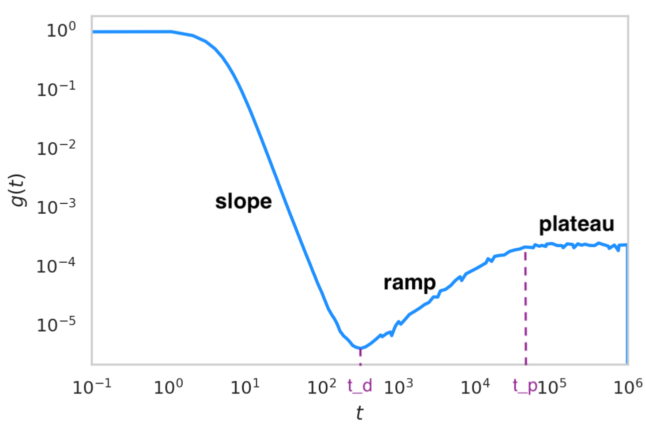

FIG. 2: An illustration of the spectral form factor vs time plot. Here t_d is the Thouless time and t_p is the Heisenberg time. Source: C. Yan, adapted from the blue curve in figure 10 of Cotler, J.S te al[3]. The figure shows the average of 90 samples of SYK model with $N = 34$ and $\beta = 5$.

If we focus on GUE, we can write the spectral form factor as

$$g(t,\beta) = \frac{\langle Z(\beta,t)Z^*(\beta,t)\rangle_{GUE}}{\langle Z(\beta)\rangle^2_{GUE}} \tag{9}$$

$$g_d(t,\beta) = \frac{\langle Z(\beta,t)\rangle_{GUE}\cdot\langle Z^*(\beta,t)\rangle_{GUE}}{\langle Z(\beta)\rangle^2_{GUE}} \tag{10}$$

$$g_c(t,\beta) = g(t,\beta) - g_d(t,\beta) \tag{11}$$

where the GUE represent the gaussian unitary ensemble[6], which can also be written as an ensemble governed by the partition function

$$\mathcal{Z}_{GUE} = \int \prod_{i,j} dM_{ij} e^{-\frac{L}{2}\mathrm{Tr}(M^2)} \tag{12}$$

We should note that this $\mathcal{Z}_{GUE}$ used to describe the ensemble that we average over rather than the energy spectrum.

The degrees of freedom in GUE are actually the eigenvalues of M and $\mathcal{Z}_{GUE}$ can be recast into an expression solely in terms of the eigenvalues λ and the eigenvalue density $\rho(\lambda)$ (or equivalently the normalized the eigenvalue density $\tilde{\rho}(\lambda)$)

$$\mathcal{Z}_{GUE} = \int D\tilde{\rho}e^{-S} \tag{13}$$

where the action is given by

$$S = -\frac{L^2}{2}\int d\lambda\tilde{\rho}(\lambda)\lambda^2 + L^2\int d\lambda_1 d\lambda_2\tilde{\rho}(\lambda_1)\tilde{\rho}(\lambda_2)\log|\lambda_1-\lambda_2| \tag{14}$$

we can see that the second term of S gives level repulsion to the eigenvalues. The saddle point of the above action is given by a semicircle

$$\tilde{\rho}_s(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2} \tag{15}$$

The eigenvalue density ρ and the action S enables us to rewrite the spectral form factor in much simpler form and thus gives an analytic result to the disconnected and connected spectral form factor. Specifically the disconnected spectral form factor to leading order in perturbative expansion in $1/L$ is given by

$$\langle Z(0,t)\rangle_{GUE} = \int_{-2}^{2} d\lambda L\tilde{\rho}_s(\lambda)e^{-i\lambda t} = \frac{LJ_1(2t)}{t} \tag{16}$$

$$g_d(t;0) = \frac{|\langle Z(0,t)\rangle_{GUE}|^2}{L^2} \sim \frac{1}{t^3} \tag{17}$$

where J is the Bessel J function. This is saying g_d gives a slope.

The connected spectral form factor is given by

$$g_c(t;0) = \int d\lambda_1 d\lambda_2 R_2(\lambda_1,\lambda_2)e^{i(\lambda_1-\lambda_2)t} \tag{18}$$

where

$$R(\lambda_1,\lambda_2) = \langle\delta\tilde{\rho}(\lambda_1)\delta\tilde{\rho}(\lambda_2)\rangle_{GUE} \tag{19}$$

$$= -\frac{\sin^2[L(\lambda_1-\lambda_2)]}{[\pi L(\lambda_1-\lambda_2)]^2} + \frac{1}{L\pi}\delta(\lambda_1-\lambda_2) \tag{20}$$

where $\delta\tilde{\rho}(\lambda) = \tilde{\rho}(\lambda) - \tilde{\rho}_s(\lambda)$. From here we can see the connected spectral form factor is a Fourier transform of the correlation between energy densities and related to the Fourier transform of energy differences. That is to say the connected spectral form factor probes energy correlations that are closer and closer to each other as time increase. This then gives

$$g_c(t;0) \sim \begin{cases} t/(2\pi L^2) & t < 2L \\ 1/(\pi L) & t \geq 2L \end{cases} \tag{21}$$

This is saying that before the Heisenberg time $t = t_p \sim L$, g_c gives a ramp and after $t = t_p \sim L$, g_c gives a plateau. Combing with the result of g_d, we know that before the Thouless time $t = t_d \sim \sqrt{L}$, the disconnected spectral form factor g_d dominates while after $t = t_d \sim \sqrt{L}$, the connected spectral form factor g_c dominates.

In particular, an intuitive interpretation of R_i is the expectation density of i-tuples. Let B be an open neighborhood

$$\int_B R_1(\lambda) d\lambda = \langle \# \text{ eigenvalues in B} \rangle$$

$$\int_{B\times B} R_2(\lambda_1,\lambda_2) d\lambda_1 d\lambda_2 = \langle \# \text{ ordered pairs of eigenvalues} \rangle$$

If we find the correlator $\langle \delta\tilde{\rho}(\lambda_1)\delta\tilde{\rho}(\lambda_2)\rangle_{GUE}$ perturbatively using the action, we can only find the ramp. This shows that the plateau is a non-perturbative effect in terms of $1/L$. More concretely,

$$\delta S = -L^2 \int d\lambda_1 d\lambda_2 \delta\tilde{\rho}(\lambda_1)\delta\tilde{\rho}(\lambda_2) \log|\lambda_1 - \lambda_2| \tag{22}$$

$$= \frac{L^2}{2}\int ds \delta\tilde{\rho}(s)\frac{1}{|s|}\delta\tilde{\rho}(-s) \tag{23}$$

so

$$\langle \delta\tilde{\rho}(\lambda_1)\delta\tilde{\rho}(\lambda_2)\rangle_{GUE} = \frac{1}{4\pi^2 L^2}\int ds e^{i(\lambda_1-\lambda_2)s}|s| + O(L^{-4}) \tag{24}$$

$$= -\frac{1}{2(\pi L(\lambda_1 - \lambda_2))^2} + O(L^{-4}) \tag{25}$$

The existence of a ramp, i.e. suppression of correlation for long distance is called spectral rigidity.

The early time behavior of the SYK model and the RMT are different, but the later time behavior are the same. A SYK Hamiltonian with N fermions in SYK corresponds to an $L \times L$ matrix in RMT, so we know that $L \sim e^N$. A perturbative expansion of RMT in terms of $1/L \sim e^{-N}$ is non-perturbative viewed as a perturbative expansion of SKY in terms of $1/N$.

IV. CONNECTION TO BLACK HOLES

According to AdS/CFT correspondence, large AdS black holes should have discrete energy spectrums[7]. This can be easily seen from the CFT perspective because of a compact boundary. But this discreteness is hard to explain from the bulk gravity side. However, given a relationship between SYK and Einstein gravity, the plateau in the spectral form factor can demonstrate the discreteness of energy spectrum. To see this, if the energy spectrum is continuous, the spectral form factor should decay indefinitely as time increases. On the other hand, if the energy spectrum is discrete as t becomes much larger than the smallest energy separation, $e^{i(E_m-E_n)t}$ in (3) gives erratic fluctuations and sum to some fix small number in late times, which gives a plateau. The long time average is given by

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt \left|\frac{Z(\beta,t)}{Z(\beta)}\right|^2 = \frac{1}{Z(\beta)^2}\sum_E N_E^2 e^{-2\beta E} \tag{26}$$

where N_E is the number of degeneracy of energy E. For simplicity if $N_E = 1$, we can write

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt \left|\frac{Z(\beta,t)}{Z(\beta)}\right|^2 = \frac{Z(2\beta)}{Z(\beta)^2} \tag{27}$$

Since $Z \sim e^{aS}$ for some positive a where S is entropy, the long time average of the spectral form factor is on the order of e^{-aS}. This is a non-perturbative effect.

[1] M. Srednicki, "Chaos and quantum thermalization," Phy. Rev. E 50, 2 (1994).

[2] E. Wigner, "Characteristic vectors of bordered matrices with infinite dimensions," Ann. Math. 62, 548 (1955).

[3] Cotler, J.S., Gur-Ari, G., Hanada, M. et al. "Black holes and random matrices," J. High Energ. Phys. 2017, 118 (2017). https://doi.org/10.1007/JHEP05(2017)118

[4] M. L. Mehta, "Random Matrices," vol. 142 of Pure and Applied Mathematics. Academic press, 3rd ed., 2004.

[5] A. Kitaev and S. J. Suh, "The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual," J. High Energy Phys. 05, 183 (2018).

[6] F. J. Dyson, "Statistical theory of the energy levels of complex systems. I," J. Math. Phys. 3, 140-156 (1962).

[7] J. M. Maldecena, "Eternal black holes in anti-de Sitter," J. High Energy Phys. 04, 021 (2003).