First-order Evolution of the Spectral Form Factor in Floquet Systems

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(Dated: June 24, 2020)

Submitted as coursework for PH470, Stanford University, Spring 2020

The spectral form factor is yet another manifestation of the connection between Random Matrix Theory (RMT) and quantum chaos. In this paper, the spectral form factor is reviewed and applied as a probe in examples of chaotic Floquet systems, with and without classical counterparts.

I. INTRODUCTION

Beginning with the pioneering work by Wigner, Random Matrix Theory (RMT) has been a fruitful diagnostic of quantum chaos. Wigner originally proposed RMT as a statistical description of the spectra of large atomic systems, whose interactions are so fascinatingly complex that they can essentially be treated as ensembles of random Hamiltonians drawn from a class respecting the symmetries of the problem. Surprisingly, such an approach works even for single-particle systems with chaotic classical analogues. In this paper, we will review a quantity, known as the spectral form factor, which has been successfully applied to probe quantum chaos in systems with and without classical analogues. Before we proceed with the main topic of Floquet systems, we first provide motivation based on time-independent systems. Given a time-independent Hamiltonian H with non-degenerate eigenspectrum $\{E_j\}$, the spectral density is

$$\rho(E) = \sum_j \delta(E - E_j) \tag{1}$$

The spectral form factor $K(t)$ is then defined as the Fourier transform of the autocorrelator of the spectral density over different "energy distances".

$$K(t) = \int e^{-\frac{i\varepsilon t}{\hbar}} \left(\left\langle \rho\left(E + \frac{\varepsilon}{2}\right) \rho\left(E - \frac{\varepsilon}{2}\right) \right\rangle_E - \bar{\rho}^2 \right) \frac{d\varepsilon}{\bar{\rho}} \tag{2}$$

where $\bar{\rho} = \langle \rho(E) \rangle_E$ and $\langle \cdot \rangle_E$ means taking the average over an energy shell centered at E of a window ΔE satisfying $\bar{\rho}\Delta E \gg 1$ such that the shell contains many energy levels. One easily obtains from the definition

$$K(t) = \frac{1}{|\mathcal{W}(E)|} \sum_{E_i, E_j \in \mathcal{W}(E)} e^{-\frac{i}{\hbar}(E_i - E_j)t} - \hbar\bar{\rho}\delta(t) \tag{3}$$

where $\mathcal{W}(E)$ is the set of energies in $\left[E - \frac{\Delta E}{2}, E + \frac{\Delta}{2}\right]$. To identify its long-term behavior, define the Heisenberg time $t_H = 2\pi\hbar\bar{\rho}$. Observe that or $t \gg t_H$, the phases $\frac{E_i t}{\hbar}$ essentially become random (recall that $\bar{\rho}$ is the average inter-energy spacing) such that for large system sizes and thus $|\mathcal{W}(E)|$, the cross-terms where $E_i \neq E_j$ become zero due to dephasing while only the cases $E_i = E_j$ contribute to the sum. Hence, $K(t) \to 1$ as $t \to \infty$. As a last remark, to allow comparison between different ensembles, the energy levels are usually rescaled such that the average spacing becomes $\bar{\rho} = 1$. The typical behavior of $K(t)$ at times larger than a time-scale known as the Thouless time t^*, after which universality is expected to set in, is depicted in Fig 1.

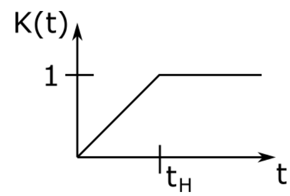

FIG. 1: Typical behavior of the spectral form factor $K(t)$ for $t \gg t^*$ where t^* is the Thouless time. In the above, t_H is the Heisenberg time.

II. RANDOM MATRIX THEORY

Now, the agenda of RMT in quantum chaos is to classify Hamiltonian ensembles into universality classes associated with predetermined random matrix ensembles based on metrics such as $K(t)$, as the system size tends to infinity. In such stochastic set-ups, we instead consider the ensemble average:

$$\bar{K}(t) = \langle K(t) \rangle_{ens} \tag{4}$$

One might think that this additional ensemble average is unnecessary as the system size is increased to infinity (intuitively, there will be more and more energies) but the spectral form factor is unfortunately not self-averaging.

The next order of business is to then compute $\bar{K}(t)$ for select random matrix ensembles which have the potential to be universality classes. Usually the considered ensembles lead to $K(t)$'s that are independent of the central energy E such that we can perform the sum in Eqn 3 over all eigenvalues to yield (dropping the δ)

$$\bar{K}(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i,j}\int e^{-i(\lambda_i-\lambda_j)t} D\lambda \tag{5}$$

where $D\lambda = P(\lambda_1, ..., \lambda_N)d\lambda_1...d\lambda_N$ is the joint distribution of $\{\lambda_j\}$ determined by the ensemble the $N\times N$ matrices are drawn from. It turns out for the well-studied General Orthogonal Ensemble (GOE) and General Unitary Ensemble (GUE),

$$\bar{K}_{GOE}(t) = \begin{cases} 2t - t\ln(1+2t) & t\leq 1 \\ 2 - t\ln\left(\frac{2t+1}{2t-1}\right) & t\geq 1 \end{cases} \tag{6}$$

$$\bar{K}_{GUE}(t) = \begin{cases} t & t\leq 1 \\ 1 & t\geq 1 \end{cases} \tag{7}$$

whose proofs[2] are rather involved. The well-known universality classes ascribed to the GOE and GUE are chaotic systems with and without time reversal symmetry respectively. However, for a Poissonian ensemble, which is expected to model integrable systems, one easily sees

$$\sum_{i,j}\int e^{-i(\lambda_i-\lambda_j)t} D\lambda = \delta_{ij} \tag{8}$$

since if $i\neq j$, the random vector $e^{-i(\lambda_i-\lambda_j)t}$ sweeps out a circle in the Argand plane with uniform probability for varying $\lambda_i - \lambda_j$, resulting in no net contribution. Thus,

$$\bar{K}_{Poisson}(t) = 1 \tag{9}$$

The different forms of $\bar{K}(t)$ hence enables us to distinguish between chaotic and integrable systems. The ensembles that describe the class of chaotic Floquet systems with and without time reversal symmetry are the Circular Orthogonal Ensemble (COE) and Circular Unitary Ensemble (CUE) whose spectral form factors[2] turn out to be the same as those of GOE/GUE for $t < t_H$,

$$\bar{K}_{COE}(t) = 2t - t\ln(1+2t) = 2t - 2t^2 + O(t^3) \tag{10}$$

$$\bar{K}_{CUE}(t) = t \tag{11}$$

III. SPECTRAL FORM FACTORS FOR FLOQUET SYSTEMS

In the context of a Floquet system which has a time-dependent albeit periodic Hamiltonian $H(t) = H(t+\tau)$, energy levels are not meaningful since the Hamiltonian is changing. As such, we redefine the spectral form factor in the following manner. Denoting U as the propagator over a single period τ, we can exploit its unitarity to decompose $U = \sum_n e^{-i\phi_n}|n\rangle\langle n|$ where $\mathcal{N} = \{\phi_n\}$ are eigenphases. The corresponding spectral density is then

$$\rho(\phi) = \frac{2\pi}{|\mathcal{N}|}\sum_n \delta(\phi-\phi_n)$$

where the leading coefficient ensures $\langle\rho\rangle = \frac{1}{2\pi}\int_0^{2\pi}\rho(\phi)d\phi = 1$. Then, we define the rescaled spectral form factor $K(t)$ for $t\in\mathbb{Z}$ as

$$K(t) \tag{12}$$

$$= \frac{|\mathcal{N}|^2}{2\pi}\int_0^{2\pi} e^{-i\theta t}\left(\left\langle \rho\left(\phi+\frac{\theta}{2}\right)\rho\left(\phi-\frac{\theta}{2}\right)\right\rangle - \langle\rho\rangle^2\right)\frac{d\theta}{\langle\rho\rangle} \tag{13}$$

$$= \sum_{\phi_i,\phi_j\in\mathcal{N}} e^{-i(\phi_i-\phi_j)t} - |\mathcal{N}|^2\delta_{t,0} \tag{14}$$

$$= tr(U^t)tr(U^{-t}) - |\mathcal{N}|^2\delta_{t,0} \tag{15}$$

where we have averaged over all eigenphases now, in anticipation that the result will be independent of the window of eigenphases considered after the subsequent ensemble average.

$$\bar{K}(t) = \langle tr(U^t)tr(U^{-t})\rangle_{ens} - |\mathcal{N}|^2\delta_{t,0} \tag{16}$$

The goal of this paper is then to review how the spectral form factors, defined by Equation 16, for specific quantum chaotic Floquet systems studied by Kos *et al.*[1] agree with Equations 10 or 11 (depending on their symmetry classes) up to first order in t. Note that the Heisenberg time in this case is $t_H = |\mathcal{N}|$.

IV. FLOQUET SYSTEMS WITH CHAOTIC CLASSICAL COUNTERPARTS

Firstly, we consider a Floquet system with a chaotic classical analog. From the path integral formulation, one obtains

$$tr(U^t) = \int_{\psi(0)=\psi(t)} e^{\frac{iS(\psi)}{\hbar}} D\psi \tag{17}$$

where the sum is performed over all paths with period t. Now, taking the semi-classical limit $h\to 0$, the main contributions to the path integral are those due to stationary actions, which precisely correspond to classical paths. Thus,

$$tr(U^t) \approx \sum_p A_p e^{i\frac{S_p}{\hbar}} \tag{18}$$

where the sum is performed over classical paths p that have period t. The coefficient $|A_p|^2$ loosely represents how long a random path tends to stay on the periodic path p and obeys the Hannay-Ozorido de Almeida sum rule[3] $\sum_p |A_p|^2 = t$ due to classical ergodicity, for times $t \ll t_H$ but large enough for ergodicity to set in. This is because periodic paths are dense in a classical ergodic system such that the sum of times spent by an arbitrary path along periodic paths must yield the total time t of that path. Thus, we have for $t \ll t_H$,

$$\bar{K}(t) = \left\langle \sum_{p,p'} A_p A^*_{p'} e^{-i\frac{S_p - S_{p'}}{\hbar}} \right\rangle_{ens} \tag{19}$$

Now, for general systems without any symmetries, only diagonal terms where $p' = p$ contribute in the above average in light of the random phase factors. Then, $\bar{K}(t) = \sum_p |A_p|^2 = t$ for $t \ll t_H$. Otherwise if the system possesses time reversal symmetry, letting $\bar{p}$ denote the time-reversed path corresponding to p, we have $A_{\bar{p}} = A_p$ and $S_p = S_{\bar{p}}$ such that for a fixed p in the above average, both $p' = p$ and $p' = \bar{p}$ contribute such that $\bar{K}(t) = \sum_p 2|A_p|^2 = 2t$. Hence, we recover the short term behaviors in Eqns 10 and 11.

V. CHAOTIC FLOQUET SPIN SYSTEM

In this section, we consider a chaotic Floquet system with no classical analog. For a spin-$\frac{1}{2}$ system with l sites, define Pauli spin operators $\{\sigma_x^{(\alpha)}\}$ where $\alpha \in \{1, 2, 3\}$ is the direction of the Pauli matrix and $x \in \{1, ..., l\}$ describes the site it acts on. Consider the periodic Hamiltonian (with unit period)

$$H(t) = H_0 + H_1 \sum_{m\in\mathbb{Z}} \delta(t - m) \tag{20}$$

where H_0 represents a many-body-localized system with l-bits $\{\sigma_x^{(3)}\}$ while H_1 represents on-site fields of strength h that act as the driving force.

$$H_0 = \sum_x J_x^1 \sigma_x^{(3)} + \sum_{x<x'} J^2_{x,x'} \sigma_x^{(3)} \sigma_{x'}^{(3)} + ... \tag{21}$$

$$H_1 = h \sum_x \sigma_x^{(1)} \tag{22}$$

Recall that the l-bits are induced by the underlying disorder hidden in H_0 such that the coefficients J_x^i's are random. Note that H exhibits time-reversal symmetry since it is real. In this scenario, the propagator over the unit period (in units $\hbar = 1$) is given by

$$U = \mathcal{T}exp\left(-i\int_0^1 H(t)dt\right) = e^{-iH_1} e^{-iH_0} \tag{23}$$

where we have used the fact that $H(t) = H_0$ for $t \in (0, 1)$ before experiencing a kick by H_1 at $t = 1$. Denote

$$V = e^{-iH_1} = \left(e^{-ih\sigma^{(1)}}\right)^{\otimes l} = v^{\otimes l} \tag{24}$$

where

$$v = e^{-ih\sigma^{(1)}} = \begin{pmatrix} \cos h & i\sin h \\ i\sin h & \cos h \end{pmatrix} \tag{25}$$

since $(\sigma^{(1)})^2 = I$. Now, let $\{|\underline{s}\rangle\}$ denote the set of 2^l joint eigenstates of $\sigma_x^{(3)}$ such that $\sigma_x^{(3)} |\underline{s}\rangle = (-1)^{s_x} |\underline{s}\rangle$ if $|\underline{s}\rangle = (s_1, s_2, ..., s_l)$ where $s_x \in \{0, 1\}$. That is, $\{|\underline{s}\rangle\}$ are the product spin up/down states with respect to the dressed operators $\{\sigma_x^{(3)}\}$. Letting $W = e^{-iH_0}$ such that $U = VW$, we have

$$W |\underline{s}\rangle = e^{-i\theta_{\underline{s}}} |\underline{s}\rangle \tag{26}$$

$$\theta_{\underline{s}} = \sum_x J_x^1 (-1)^{s_x} + \sum_{x<x'} J^2_{x,x'} (-1)^{s_x + s_{x'}} + ... \tag{27}$$

Meanwhile, the matrix elements of V with respect to $\{|\underline{s}\rangle\}$ factorize due to the tensor product structure of V.

$$\langle \underline{s}|V|\underline{s}'\rangle = \prod_{i=1}^{l} v_{s_x, s'_x} \tag{28}$$

Armed with these relationships, we are now ready to compute the spectral form factor by repeatedly inserting identities $\sum_{\underline{s}_\tau} |\underline{s}_\tau\rangle \langle \underline{s}_\tau| = 1$. Firstly,

$$tr(U^t) = tr(VWVW...) \tag{29}$$

$$= \sum_{\underline{s}_1,...,\underline{s}_t} \langle \underline{s}_1|VW|\underline{s}_2\rangle \langle \underline{s}_2|VW|\underline{s}_3\rangle ... \langle \underline{s}_t|VW|\underline{s}_1\rangle \tag{30}$$

$$= \sum_{\underline{s}_1,...,\underline{s}_t} e^{-i\sum_{\tau=1}^t \theta_{\underline{s}_\tau}} \prod_{x=1}^{l} \prod_{\tau=1}^{t} v_{s_{x,\tau}, s_{x,\tau+1}} \tag{31}$$

where we have identified $t + 1$ with 1 such that

$$\bar{K}(t) = \sum_{\underline{s}_1,...,\underline{s}_t} \sum_{\underline{s}'_1,...,\underline{s}'_t} \left\langle e^{-\mathrm{i}\sum_{\tau=1}^t \left(\theta_{s_\tau} - \theta_{s'_\tau}\right)} \right\rangle_{ens} \times \prod_{x=1}^{\ell} \prod_{\tau=1}^{t} v_{s_{x,\tau}, s_{x,\tau+1}} v^*_{s'_{x,\tau}, s'_{x,\tau+1}} \tag{32}$$

Now, to leading order, due to the random phases in the ensemble average, we have

$$\left\langle e^{-\mathrm{i}\sum_{\tau=1}^{t}\left(\theta_{s_\tau}-\theta_{s'_\tau}\right)}\right\rangle = \delta_{\langle \underline{s}_1,\ldots,\underline{s}_t\rangle,\langle \underline{s}'_1,\ldots,\underline{s}'_t\rangle} \tag{33}$$

where $\langle \underline{s}_1,\ldots,\underline{s}_t\rangle$ represents a lexicographically ordered string of words $\underline{s}_1,...,\underline{s}_t$. Note that the Kronecker-delta is between entire sorted lists since we only need $\sum_\tau^t \theta_{s_\tau} = \sum_\tau^t \theta_{s'_\tau}$ for the phases to cancel. Finally, for $t \ll t_H = 2^l$, we can also see that the dominant number of strings $\underline{s}_1, \underline{s}_2, ..., \underline{s}_t$ are such that no two words are repeated, by elementary combinatorics. Thus, to obtain the leading order effect, we just have to consider the cases where there exist some $\pi \in S_t$ (S_t being the permutation group acting on t elements) such that $\underline{s}'_\tau = \underline{s}_{\pi(\tau)}$. Thus, up to a $O\left(\frac{t}{2^l}\right)$ correction,

$$\bar{K}(t) = \sum_{\pi\in S_t} Z_\pi^l \tag{34}$$

where

$$Z_\pi = \sum_{s_1\ldots s_t}\prod_{\tau=1}^{t} v_{s_\tau,s_{\tau+1}} v^*_{s_{\pi(\tau)},s_{\pi(\tau+1)}} \tag{35}$$

Now, it turns out that to yet another $O\left(\frac{t}{2^l}\right)$ correction, we just have to consider $\pi \in S_t$ such that for all $s_1, .., s_t$,

$$\prod_{\tau=1}^{t} v_{s_\tau,s_{\tau+1}} = \prod_{\tau=1}^{t} v^*_{s_{\pi(\tau)},s_{\pi(\tau+1)}} \tag{36}$$

These are precisely the cyclic permutations $\pi : \tau \to \tau + k$ for some $0 \le k \le t-1$ and anti-cyclic permutations $\pi : \tau \to t+1-\tau-k$ for some $0 \le k \le t-1$ for a total of $2t$ such permutations. In these cases, Z_π can evaluated in an Ising-like fashion by defining the transfer matrix $T_{ss'} = |v_{ss'}|^2$.

$$T = \begin{pmatrix}\cos^2 h & \sin^2 h\\ \sin^2 h & \cos^2 h\end{pmatrix} \tag{37}$$

Then, applying Eqn 36,

$$\begin{aligned} Z_\pi &= \sum_{s_1\ldots s_t}\prod_{\tau=1}^{t} |v_{s_\tau,s_{\tau+1}}|^2 \\ &= \sum_{s_1\ldots s_t} \langle s_1|T|s_2\rangle \langle s_2|T|s_3\rangle \ldots \langle s_t|T|s_1\rangle \\ &= Tr(T^t) \\ &= 1 + (\cos 2h)^t \end{aligned}$$

where we have used the fact that the eigenvalues of T are 1 and $\cos 2h = \cos^2 h - \sin^2 h$ in computing $Tr(T^t)$. Hence,

$$\bar{K}(t) = 2t\left[1 + (\cos 2h)^t\right]^l + O\left(\frac{t}{2^l}\right) \tag{38}$$

$$\approx 2t \quad \text{for} \quad t \gg t^* \tag{39}$$

with the Thouless time being

$$t^* = -\frac{\ln l}{\ln\cos 2h} \tag{40}$$

which agrees with Eqn 10. It should be remarked that for higher-order corrections, one needs to perform a careful accounting[1] of other permutations π as well as other strings $\underline{s}_1, ..., \underline{s}_t$ with duplicates.

The numerous approximations made in this section are summarized as: 1) the same-phase approximation in Eqn 33 which ignored fluctuations, 2) the restriction to strings with no repeated words in Eqn 34 and, 3) the restriction to strings obtained from cyclic and anti-cyclic permutations of distinct words.

VI. CONCLUSION

All-in-all, we have demonstrated how the spectral form factor can be an adept probe of quantum chaos in Floquet systems with and without classical counterparts.

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2 *M. L. Mehta, Random Matrices and the Statistical Theory of Spectra, 2nd ed. (Academic, New York, 1991).*

3 *J. H. Hannay, A. M. Ozorio De Almeida, Periodic orbits and a correlation function for the semiclassical density of states, J. Phys. A: Math. & Gen.* ***17****, 3429 (1984).*