Operator Spreading, Out-of-Time-Correlators and Entanglement Growth in One Dimensional Spin Chains

Jun Ho Son
Department of Physics, Stanford University, Stanford, CA 94305
(Dated: June 13, 2020)

Submitted as coursework for PH470, Stanford University, Spring 2020

Here, I review characteristic behaviors of operator spreading, out-of-time correlators, and entanglement growth in three different models defined on one dimensional spin chains: the Haar-random unitary circuits, the kicked Ising model, and the fractal Clifford circuits. In the Haar-random unitary circuits, mapping from the problem of studying the operator spreading to the random walk allows one to derive many results analytically. The analytical results indicate that operators spread hydrodynamically with velocity and diffusion constant set by microscopic details. Using this result, one can also derive that entanglement entropy grow linearly and that out-of-time correlators grows sharply, albeit not exponentially, in the regime set by the velocity of the operator hydrodynamics. I discuss how the similar behaviors are manifest in the kicked Ising model. This leads to the conjecture that the behaviors shown in the random unitary circuits are universal in a wide class of one-dimensional models. Meanwhile, the fractal Clifford circuit provides an example of a one-dimensional system whose behaviors are significantly different from the ones of the two aforementioned models although operators do spread in the late time in the fractal Clifford circuit as well.

I. INTRODUCTION

In recent years, there has been great interest in how to characterize quantum chaos. This is certainly an interesting and important problem that had not been in limelight for a while. However, its recently discovered relation to the black hole physics and the progress in understanding the physics of the many-body localized systems and non-equilibrium systems brought the subject its deserved attention.

In the classical mechanics, the terminology that characterizes the key feature of chaotic feature is the butterfly effect. A small difference in the initial condition leads to exponentially large difference in outcome in late time – this is how a snap of wings that a butterfly makes in Beijing causes hurricane on the other side of the world. In more mathematical language, such extreme sensitivity to the initial conditions can be expressed as the following:

$$\left|\frac{\partial x(t)}{\partial x_0}\right|^2 = |\{x(t), p(0)\}|^2 \sim e^{2\lambda t} \tag{1}$$

Here, the bracket refers to the Poisson bracket; x_0 refers to the initial condition; λ is an exponent called Lyapunov exponent. In the classical chaotic systems, the extreme dependence on initial conditions and the existence of the Lyapunov exponent that characterizes the sensitivity is a statement *established in mathematical sense.*

How can we characterize quantum chaos? One way is to generalize the classical picture by applying a naive canonical quantization, i.e., replacing position and momentum with the corresponding Hermitian operators and replacing the Poisson bracket with commmutators. Applying this to the rhs of the Eq. (1) lead to quantities dubbed the out-of-time correlators $\mathcal{C}_{V,W}(t)$:

$$\mathcal{C}_{V,W}(t) = \langle [V(t), W][V(t), W]^\dagger \rangle \tag{2}$$

The stream of thoughts that leads to the out-of-time correlators as the relevant quantities in the diagnosis of quantum chaos was admittedly naive. However, it is shown that in systems such as black holes and in the SYK model (for example, see[1] and[2]), the out-of-time correlators do grow exponentially in the certain time regime with the Lyapunov exponents characteristic to the systems.

As for the quantities that are more inherent to quantum mechanics, one can track how a localized operator, in the Heisenberg picture, spreads out in the late time. The process of a localized operator evolving into an operator having a global effect on a system captures the notion of quantum scrambling. Alternatively, one may harness some of the quantities in quantum information theory. In particular, the time evolution of the von Neumann entropy

$$S_{1,A}(t) = -\mathrm{Tr}[\rho_A(t) \log \rho_A(t)] \tag{3}$$

and the Renyi entropy, occasionally computationally more convenient than the von Neumann entropy:

$$S_{n\geq 2,A}(t) = \frac{1}{n-1} \log \mathrm{Tr}[\rho_A^n(t)] \tag{4}$$

characterizes how degrees of freedom in the subsystem A become strongly entangled with the rest of degrees of freedom in the system. The increase in the entanglement entropy indicates that the larger degrees of the "spooky action in the distance", leading to the dramatic global effect on the system even if the system might be modified locally at the early time.

There are indeed good pictures of how the quantities mentioned so far should be related to quantum chaos. However, contrary to the classical cases in which the existence of Lyapunov exponents characterize how classically chaotic systems behave, there is no *a priori* statement about what is the universal behavior of these quantities in quantum chaotic systems or whether even there is such a universal behavior. For example, when we are considering the systems with the concept of locality baked in (for example, any local lattice models in one dimension or higher), it is known that Lyapunov exponent is a tricky concept to define[3].

The goal of this paper is to review the contents of von Keyserlingk et al. [4] (there is a closely related paper[5] with some complementary viewpoints and results in higher dimension as well). In von Keyserlingk et al. [4], the above quantities in three different models of one-dimensional quantum spin chains are studied. Studying these model shed light on some of the questions about the diagnostics of quantum chaos. While none of the systems they studied have exponential increase in out-of-time correlators as expected from classical systems and as in some of quantum system, there exists sharp increase in out-of-time correlators and entanglement entropy in the two of the systems. Interestingly, many of such behaviors can be traced from the fact that operators spread in *hydrodynamical manner*. Meanwhile, the other system provide the counterexample in which the quantities of our interest behave differently from the two models.

In Sec. II, the basic setup and the three models analyzed in the paper − the Haar-random unitary circuits, the kicked Ising models, and the fractal Clifford circuits are introduced. In Sec. III, I will give a detailed introduction to the analytical result in the random unitary circuits. I will finish with how the behaviors derived from the random unitary circuits apply to the kicked Ising models and the fractal Clifford circuits in Sec. IV.

II. MODELS AND METHODOLOGY

In this section, I specify the models and the quantities that characterize scrambling and chaos studied in[4]

A. Basic Setup

The basic setup common to the all three models is that we start with a one-dimensional chain of length L, and at each site we have a clock variable with q possible states. I will introduce some notations and facts that would be helpful in the future discussion. The two basic one-site operators that act on the clock variable at site i (with $-\frac{L}{2}+1 \leq i \leq \frac{L}{2}$. I am taking $i=0$ to be at the center of the chain to make the bookkeeping easier later in the paper) are:

$$Z_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{\frac{2\pi i}{q}} & 0 & \cdots & 0 \\ 0 & 0 & e^{\frac{4\pi i}{q}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\frac{2(q-1)\pi i}{q}} \end{pmatrix} \tag{5}$$

$$X_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6}$$

Note that $X_i Z_i = e^{2\pi i/q} Z_i X_i$. X_i rotate clock variables, and Z_i "signals" which clock variable state at site i is in by returning different q root of unities according to the current clock variable state. Note that X_i and Z_i are natural generalizations of Pauli matrices σ^x and σ_z at $q=2$ to other integers.

Any one site operator can be represented as a linear combination of

$$O_{\mu_i} = Z_i^{\mu_i^{(Z)}} X_i^{\mu_i^{(Z)}}, \tag{7}$$

$\mu_i = (\mu_i^{(Z)}, \mu_i^{(X)})$ being a pair of two $\mathbb{Z}_q$ numbers. In other words, $\{O_{\mu_i}\}$ forms an orthonormal basis of the linear space of one-site operators. Similarly, one may define

$$\mu = (\mu_{-\frac{L}{2}+1}, \mu_{-\frac{L}{2}+2}, \mu_{-\frac{L}{2}+3}, \cdots \mu_{\frac{L}{2}}) \tag{8}$$

μ is an array of μ_i for all possible i's, and each μ_i, as before, is a pair of two $\mathbb{Z}_q$ numbers $\mu_i^{(Z)}$ and $\mu_i^{(X)}$. Then, one can define

$$O_\mu = \bigotimes_{i=-\frac{L}{2}+1}^{\frac{L}{2}} O_{\mu_i} \tag{9}$$

$\{O_\mu\}$ forms an orthonormal basis of all operators on the 1D chain. I will occasionally refer O_μ as a Pauli string because it is a direct product of the generalized Pauli matrices.

A choice of models refers to a choice of time evolution operators of the 1D spin chain. In this work, time evolution operators are defined by quantum circuits, each circuit element representing a *local* unitary transformation that gives time evolution of a state after a discrete time step.

From the given time evolution, I would be interested in accessing the following quantities to characterize quantum chaos:

1. $\rho_R(s,t)$: $\rho_R(s,t)$ is a function of a spatial index s and a time t that contains information about how the right end of an operator $Z_{i=0}(t)$ evolves. To define this quantity more precisely, let me write the Heisenberg representation of Z_0 at time t as:

$$Z_0(t) = \sum_\nu a_\nu(t) O_\nu \tag{10}$$

$\{a_\nu(t)\}$ contains the full information about how Z_0 evolves in time. Additionally, define

$$\eta(\mu) = (\min i \text{ that satisfies } \mu_j = (0,0) \text{ for any } j \geq i\ ) \tag{11}$$

Note that O_μ acts trivially on all sites on the right of $\eta(\mu)$. Physically, $\eta(\mu)$ denotes "the right end" of the operator O_μ. Hence, once I define $\rho_R(s,t)$:

$$\rho_R(s,t) = \sum_\nu |a_\nu(t)|^2 \delta_{\eta(\nu)=s} \tag{12}$$

it is clear that $\rho_R(s,t)$ quantifies how the right end of the time-evolved operator $Z_0(t)$ behaves.

2. The out-of-time correlator; more specifically,

$$\mathcal{C}(s,t) = \mathcal{C}_{Z_s,Z_0}(t) \tag{13}$$

3. The growth of entanglement entropy and Renyi entropy $S_{n,A}(t)$, taking the initial state at $t=0$ to be a product state.

B. The Haar-Random Unitary Circuit

The time evolution in this model can be written as:

$$|\psi(t+1)\rangle = \left(\prod_j U^{(2)}_{j,t}\right)\left(\prod_j U^{(1)}_{j,t}\right)|\psi(t)\rangle \tag{14}$$

where $U^{(2)}_{j,t}$'s($U^{(1)}_{j,t}$'s) are Haar-random $U(q^2)$ matrices acting on site $2j-1$ and $2j$ (site $2j$ and $2j+1$) respectively? . One may imagine the time evolution of the model as a set of unitary local transformations drawn in Fig. 1, each rectangle corresponding to a $U(q^2)$ Haar-random matrix.

One important consequence of the geometry shown in Fig. 1 is the existence of the light-cone. A operator localized at a single site cannot spread beyond the light-cone illustrated as two dotted lines in Fig. 1. When starting from the operator localized at the site $i=0$, the light-cone corresponds to the region between $s=t$ and $s=-t$ lines. The light-cone defines a natural "speed of light" $v_{LC}=1$.

Remarkably, we will see in the next section that many of the quantities that we are interested in can be computed *analytically* in this setup.

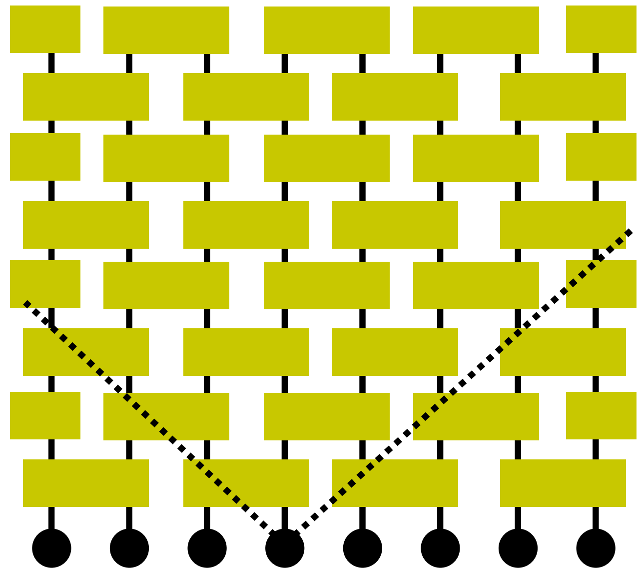

FIG. 1: The arrays of local unitary circuits and the light cone in the Haar-random unitary circuits and the kicked Ising model

C. The Kicked Ising Model

The time evolution of the Kicked Ising model is given by the following time-dependent Hamiltonian with $q=2$:

$$H(t) = \begin{cases} \sum_i Z_i Z_{i+1} + h\sum_i Z_i & 0<t<\frac{T}{2} \\ g\sum_i X_i & \frac{T}{2}<t<T \end{cases},\ H(t) = H(t+T) \tag{15}$$

The time evolution of the system can be alternatively written as the following series of local unitary transformations:

$$\mathcal{T}e^{-i\int_0^T H(t)dt} = \left(\prod_j U_4^{(j)}\right)\left(\prod_j U_3^{(j)}\right)\left(\prod_j U_2^{(j)}\right)\left(\prod_j U_4^{(j)}\right) \tag{16}$$

With

$$\begin{aligned} U_4^{(j)} &= e^{i\frac{gT}{4}(X_{2j}+X_{2j+1})} \\ U_3^{(j)} &= e^{i\frac{gT}{4}(X_{2j-1}+X_{2j})} \\ U_2^{(j)} &= e^{i\frac{T}{2}(Z_{2j}Z_{2j+1}+hZ_{2j}+hZ_{2j+1})} \\ U_1^{(j)} &= e^{i\frac{T}{2}(Z_{2j-1}Z_{2j})} \end{aligned} \tag{17}$$

Note that $U_1^{(j)}$ and $U_3^{(j)}$ are two-site local unitary gates acting on the site $2j-1$ and $2j$; similarly, $U_2^{(j)}$ and $U_4^{(j)}$ are two-site local unitary gates acting on the site $2j$ and $2j+1$. Hence, this model may be understood as a special example of unitary circuits with the geometry in Fig. 1. Hence, the statement about the existence of the light-cone applies here as well.

In this model, there are few things that can be figured out analytically. However, using the algorithm called Time-Evolving Block Decimation (TEBD)[6], a powerful tool that allows one to compute time evolution of 1D chains based on the tensor network formalism, one can compute many quantities of our interest numerically. This will allow comparison to the analytical results obtained from the Haar-random unitary circuits.

D. The Fractal Clifford Circuits

Clifford circuits are defined as unitary circuits with $q = 2$ whose unitary time evolution operators of the whole system are in the Clifford subgroup of the full unitary group. The Clifford group is defined as a subgroup of the full unitary group that maps each Pauli string to a different Pauli string, rather than to a linear combination of Pauli strings as elements of the full unitary group generically do.

There is vast literature on Clifford circuits, especially when they are translationally invariant and go by the name Clifford quantum cellular automata. Hence, one can borrow some insight from the already existing literature to analyze them. In particular, it is known that in one-dimensional translationally invariant Clifford circuits, the time evolution of Pauli strings reduces to either of these three behaviors: (i) periodic in time (ii) time-glide or (iii) fractal[7]. If the operator evolution is periodic in time or time-glide, there is clearly no operator spreading or scrambling of information, and such choice is not suitable for the purpose of the study.

Meanwhile, the time evolution given by a Clifford circuit can be "fractal" in the following sense: One can track the time evolution of a localized Pauli string by marking each site at time t when the time-evolved Pauli string at time t acts non-trivially on the sites. Then, the marking of the sites as t increases have a fractal pattern. In such time evolution, there is clearly a notion of operator spreading and scrambling. Hence, the fractal Clifford circuits are the natural types of Clifford circuits to study if one is interested in scrambling and quantum chaos. Recall that the classification of Clifford circuits I mentioned heretofore is only established for translationally invariant circuits. Hence, by saying that I am interested in the fractal Clifford circuits, I am restricting myself to translationally invariant circuits.

Finally, I want to make a remark that Clifford circuits can be efficiently simulated on *classical computers*[8] despite being quantum circuits. Hence, when one wants to compute quantities that no theorems from quantum information theory provides guidance, one can always resort to numerics.

III. ANALYTICAL RESULTS FROM THE RANDOM UNITARY CIRCUITS

In this section, I will review analytical results derived from the Haar-random unitary circuit. The key result is in Sec. III A where the time evolution of $\rho_R(s,t)$ is mapped to *biased random walk* and the analytically exact formula for $\rho_R(s,t)$ can be obtained. Out-of-time correlators and $n = 2$ Renyi entropies can be also computed from the quantity $\rho_R(s,t)$.

A. Analytical Results on $\rho_R(s,t)$

We will be primarily interested in computing $\overline{\rho_R(s,t)}$, the overline denoting average over all possible random unitaries. Note that by definition,

$$\overline{\rho_R(s,t=0)} = \delta_{s=0} \tag{18}$$

To study $\overline{\rho_R(s,t\neq 0)}$, let me study in detail how the random unitary transformation acting on site s and $s+1$ induces the recursion relation between $\overline{\rho_R(s,t)}$ at different time.

Pauli strings that contribute to $\rho_R(s,t)$ acts non-trivially on s but trivially on $s+1$. Also, there are $q^4 - 1$ possible combinations of pauli matrices that act on s or $s+1$ non-trivially (Note that the condition "non-trivially" explicitly excludes identity matrix, hence -1 to q^4). Among them, $q^2 - 1$ choices act trivially on $s+1$; $q^2(q^2-1)$ choices act non-trivially on $s+1$. Thanks to the Haar-random distribution of the unitary gates, *upon averaging over all unitary transformations, Pauli strings that contribute to $\rho_R(s,t)$ are transformed to any of q^4-1 Pauli strings that act non-trivially on either site s or $s+1$ with equal weight.* Hence, $\overline{\rho_R(s,t)}$ contributes to $\overline{\rho_R(s,t+1)}$ and $\overline{\rho_R(s+1,t+1)}$ as:

$$\begin{aligned}\overline{\rho_R(s,t+1)} &= \frac{1}{q^2+1}\overline{\rho_R(s,t)} + \text{other dependence} \\ \overline{\rho_R(s+1,t+1)} &= \frac{q^2}{q^2+1}\overline{\rho_R(s,t)} + \text{other dependence}\end{aligned} \tag{19}$$

One can similarly show how $\overline{\rho_R(s+1,t)}$ is related to $\overline{\rho_R(s,t+1)}$ and $\overline{\rho_R(s+1,t+1)}$. Additionally, since the unitary transformation only acts on s and $s+1$, it is impossible for ρ_R's at other sites contribute to $\overline{\rho_R(s,t+1)}$ and $\overline{\rho_R(s+1,t+1)}$. Hence, one can show that a random two-site unitary transformation acting on site s and $s+1$ at time t induces the following recursion relation:

$$\begin{aligned}\overline{\rho_R(s,t+1)} &= \frac{1}{q^2+1}\left(\overline{\rho_R(s,t)} + \overline{\rho_R(s+1,t)}\right) \\ \overline{\rho_R(s+1,t+1)} &= \frac{q^2}{q^2+1}\left(\overline{\rho_R(s,t)} + \overline{\rho_R(s+1,t)}\right)\end{aligned} \tag{20}$$

Considering the geometry of the circuits in which unitary transformation act on sites in an alternating manner, introducing

$$\tilde{\rho}_R(x,t) = \overline{\rho_R(2x-1,2t)} + \overline{\rho_R(2x,2t)} \tag{21}$$

$\tilde{\rho}_R(x,t)$ obeys the following local recursion relation:

$$\tilde{\rho}_R(x,t+1) = p_1\tilde{\rho}_R(x-1,t)+p_2\tilde{\rho}_R(x,t)+(1-p_1-p_2)\tilde{\rho}_R(x+1,t) \tag{22}$$

where p_1, p_2 is defined as:

$$p_1 = \frac{2q^2}{(q^2+1)^2},\ p_2 = \frac{q^4}{(q^2+1)^2} \tag{23}$$

By deriving this recursion relation, we mapped the computing $\tilde{\rho}_R(x,t+1)$ to that of the random walk – an object at the coordinate x and time t, after unit time, has p_1 probability to move to the right, p_2 probability to stand still, and $(1-p_1-p_2)$ probability to move to the left. Note that the probability to move to the right is large than the probaiblity to move to the left – as expected how the right end of the time-evolved operator move.

One can find an analytical expression satisfying the above recursion relation and the initial condition Eq. (18). It is given by:

$$\tilde{\rho}_R(x,t) = \frac{q^{2(t+x)}}{(1+q^2)^{2t}}\frac{(2t)!}{(t+x)!(t-x)!} \tag{24}$$

While the above expression and various approximations of the above result can be useful in deriving analytical formulae for other quantities, a particularly insightful expression in this subsection is obtained by taking the limit $x,\ t \to \infty$ with $\frac{x}{t} \approx v_B = \frac{q^2-1}{q^2+1}$ nearly fixed. In this limit,

$$\tilde{\rho}_R(x,t) \sim e^{-\frac{(x-v_B t)^2}{(1-v_B^2)t}} \tag{25}$$

The following are the key lessons from the above expression:

- The right end of the operator, roughly speaking, moves with the group velocity $v_B = \frac{q^2-1}{q^2+1}$. The two important features about v_B is that it depends on microscopic detail (in the case of the random unitary circuits, q) rather than is fixed to some universal constant and that it is generically smaller than the light-cone velocity $v_{LC} = 1$
- the right end of the operator has a Gaussian shape whose width $W \sim \sqrt{t}$

The above bullet points are illustrated in Fig. 2

B. Comment on Hydrodynamics

Here, I briefly comment on how the time evolution of $\rho_R(s,t)$ can be connected to hydrodynamics. Hydrodynamical behaviors arise when there are continuous symmetries and therefore the associated conserved quantities.

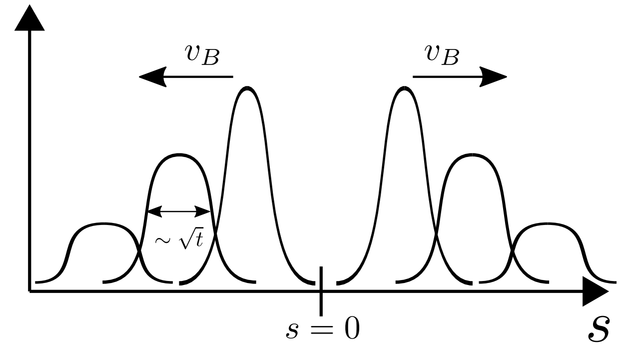

FIG. 2: The evolution of the operator "wavefront" as indicated by Eq. (25). The two ends of the operators move with butterfly velocity v_B; the both ends are "fuzzed" in the shape of the Gaussian function, whose width scales like $\sqrt{t}$.

Hence, the natural macroscopic variable is the density of the conserved quantity $\rho(\vec{x},t)$. Then the conservation law

$$\partial_t \rho(\vec{x},t) + \vec{\nabla}\cdot\vec{J}(\vec{x},t) = 0 \tag{26}$$

along with the constitutive relation that relates the conserved current $\vec{J}(\vec{x},t)$ to the local density associated with local density:

$$\vec{J}(\vec{x},t) \sim \alpha\vec{f}(\vec{x},t)\rho(\vec{x},t) + \beta\nabla\rho(\vec{x},t) + \cdots \tag{27}$$

The usual assumption is that the perturbation of the macroscopic variables from the equilibrium values is long-wavelength and small enough that the derivative expansion is valid. Hence the dots in the above equations contain terms higher in the derivative that are expected to be relatively unimportant. The coefficients α and β are determined by the microscopic details. Hydrodynamics arises in (of course, as the name suggest) fluid dynamics and electronic transport in graphene[9].

The statement that the operator spreading is hydrodynamical in the random unitary circuit is somewhat counter-intuitive because there is no conserved quantity in the system; note that even energy is not conserved quantity in the model. However, note that

$$\sum_x \rho_R(x,t) = 1 \tag{28}$$

Hence, $\tilde{\rho}_R(x,t)$ can be thought as density of the emergent conserved quantity $\sum_x \rho_R(x,t)$. Then, invoking the constitutive relation

$$J(x,t) = v_B\tilde{\rho}_R(x,t) + \frac{\sqrt{1-v_B^2}}{2}\partial_x\tilde{\rho}_R(x,t) + \cdots \tag{29}$$

gives the behavior found in in Eq. (25).

C. Out-of-time correlators and Entanglement entropies

One can also obtain analytical expression for the random-unitary-averaged out-of-time correlators and $n = 2$ Renyi entropy when the random average is taken as[?] :

$$\overline{S}_{2,A}(t) = -\log \overline{e^{-S_{2,A}(t)}} \tag{30}$$

i.e., taking random-unitary-average of $e^{-S_{2,A}(t)}$ and taking the logarithm.

However, both the derivations and the expressions are quite complicated and technical. Hence, I chose to simply summarize the key features of the results here rather than explicitly working through the derivations. The out of time correlators and the $n = 2$ entropy exhibit the following behaviors:

- Outside the light-cone, i.e. $s > t$, the out-of-time correlator $\mathcal{C}(s,t) = 0$. However, as one gets inside the light-cone, $\mathcal{C}(s,t)$ acquires non-zero value and start to increase. $\mathcal{C}(s,t)$ experience sharp increase near $s = v_B t$, but the increase is neither exponential nor characterized by the Lyapunov exponents as in some of quantum systems are shown to be. After the sharp increase, , deep inside the light-cone, $\mathcal{C}(s,t)$ saturates to 1.

- The Renyi entropy grows linearly with a slope $v_E < v_B$ then saturates to an equilibrium value. The reason that the speed of the entanglement growth is slower than v_B is attributed to the Gaussian broadening of $\rho_R(x,t)$.

IV. COMPARISON TO THE OTHER MODELS

In this section, I will discuss how the behaviors derived in the previous section is manifest or absent in the kicked Ising model and the fractal Clifford circuits. The original paper[4] presents numerical studies of the kicked Ising model using TEBD, focusing on $\rho_R(s,t)$ and out-of-time correlators. The numerical results indicate that the behaviors summarized at the end of each subsection in the previous section are manifest in the kicked Ising model as well.

It would be more insightful to discuss the behaviors of out-of-time correlators and $\rho_R(s,t)$ in more detail. Similarly to the Haar-random unitary circuits, the behaviors of out-of-time correlators and $\rho_R(s,t)$ allow them to set the velocity v_B at which the local operators spread. In the case of the kicked Ising model, similarly to the random unitary circuits, v_B is smaller than the light-cone velocity, but v_B is *not equal to* $\frac{q^2-1}{q^2+1}$ (recall that in the kicked Ising model, $q = 2$): Instead, v_B is some non-universal number that can be tuned by changing the x-field strength g. This suggests the point I made in the earlier section that v_B is *not universal* and dependent on microscopic details. Also, the sharp increase of out-of-time corrrelator around $s \approx= v_B t$ is also observed.

Additionally, they study the width of the Gaussian wavefront manifest in Eq. (25) by studying

$$\sigma(t) = \sqrt{\sum_s s^2 \rho_R(s,t) - \left(\sum_s \rho_R(s,t)\right)} \tag{31}$$

and extract that $\sigma(t)$ has a power law dependence $\sim t^\alpha$ with α close to 0.5. Note that the kicked Ising model is spatial-translation-invariant and discrete time-translation-variant, while the random unitary circuits break all the symmetries present in the kicked Ising model. However, the fact that the kicked Ising model shows similar behaviors suggest that the behaviors we derived in the earlier section are *universal in a large class of 1D systems*. Such "universality class" seems to be distinct from the ones that SYK models and black holes belong to since the 1D models studied here do not have a clearly defined Lyapunov exponent from out-of-time-correlators. Finally, the paper noted that there is a recent result where the similar hydrodynamic behaviors are observed in the time evolution from the static Hamiltonian as well[10], further supporting the idea of the universality class of quantum chaos.

Meanwhile, $\rho_R(s,t)$ and out-of-time correlators behave very anomalously in the fractal Clifford circuits. Recall that the fractal Clifford circuits map Pauli strings to different Pauli strings under time evolution. Hence, $\rho_R(s,t)$ is always fixed to be 1 or 0. Since the fractal nature of the time evolution indicates that the length of Pauli strings do increase on average upon time evolution, it is natural to expect that asymptotically the point in the spacetime where $\rho_R(s,t) = 1$, the "operator wavefront", moves toward $s, t \to \infty$. However, there will be no Gaussian broadening of the wavefront.

Similarly, $Z_0(t)$ will be Pauli strings that in general satisfy

$$[Z_0(t), Z_s] = (1 - e^{2\pi i f(s,t)/q} Z_0(t) Z_s \tag{32}$$

where $f(s,t)$ is some function that gives $\mathbb{Z}_q$ number. Note that due to the fractal nature of the time evolution, there is generically no guarantee for $f(s,t)$ to be fixed in late time; in fact, $f(s,t)$ will oscilliate in the late time as well, so do out-of time correlators. This is in sharp contrast to the saturation of out-of-time correlators to constant values at the late time in the random unitary circuits and kicked Ising model.

Finally, I would like to point out that for certain classes of initial states, it is shown mathematically that the entanglement grows linearly with time[7]. Also, in the original paper there are some addtional numerical studies that observe the linear growth of the entanglement entropy in more general classes of initial states. Combined with the earlier statement on $\rho_R(s,t)$, one can conclude that there is certainly some notion of scrambling and chaos in the fractal Clifford circuits as well. However, their behaviors seem to be significantly different from the ones expected from the "universality class" that the Haar-random unitary circuits and the kicked Ising model belong to.

---

[1] J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016), URL https://link.aps.org/doi/10.1103/PhysRevD.94.106002.

[2] J. Maldacena, S. H. Shenker, and D. Stanford, Journal of High Energy Physics **2016**, 106 (2016).

[3] V. Khemani, D. A. Huse, and A. Nahum, Phys. Rev. B **98**, 144304 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.144304.

[4] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Phys. Rev. X **8**, 021013 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.021013.

[5] A. Nahum, S. Vijay, and J. Haah, Phys. Rev. X **8**, 021014 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.021014.

[6] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003), URL https://link.aps.org/doi/10.1103/PhysRevLett.91.147902.

[7] J. Gütschow, Applied Physics B **98**, 623 (2010).

[8] D. Gottesman, arXiv preprint quant-ph/9807006 (1998).

[9] A. Lucas and K. C. Fong, Journal of Physics: Condensed Matter **30**, 053001 (2018), URL https://doi.org/10.1088%2F1361-648x%2Faaa274.

[10] E. Leviatan, F. Pollmann, J. H. Bardarson, D. A. Huse, and E. Altman, arXiv preprint arXiv:1702.08894 (2017).

[11] Mathematically, choosing Haar-random unitary matrices is essentially tantamount to choosing matrices from uniform distribution defined with Haar measure. Haar-random is an only mathematically sensible way to define uniform distribution in the space of unitary matrices.

[12] There can be concerns of whether the $n = 2$ Renyi entropy results capture the behavior of the von Neumann entanglement entropy and whether the particular random averaging is valid. The original paper presents some numerical evidences that address these concerns